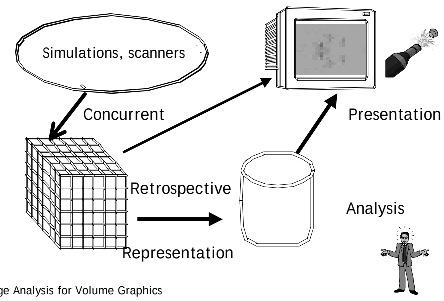
Wavelets

Raghu Machiraju, The Ohio State University

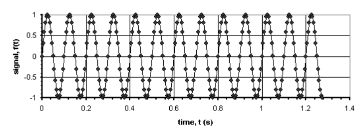
State-Of-Affairs

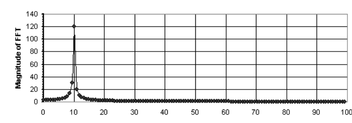


Why Wavelets?

- We are generating and measuring larger datasets every year
- We can not store all the data we create (too much, too fast)
- We can not look at all the data (too busy, too hard)
- We need to develop techniques to store the data in better formats

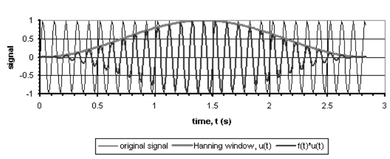
Data Analysis

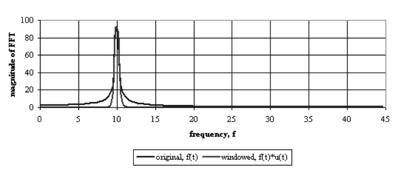




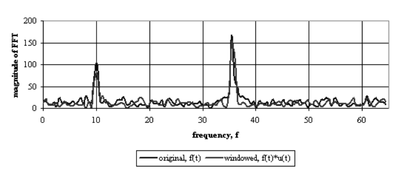
- Frequency spectrum correctly shows a spike at 10 Hz
- Spike not narrow - significant component at between 5 and 15 Hz.
- Leakage - discrete data acquisition does not stop at exactly the same phase in the sine wave as it started.

QuickFix





Windowing & Filtering



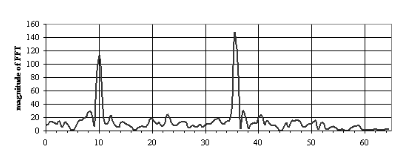
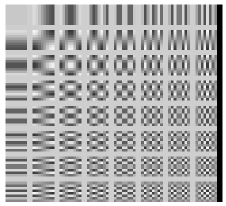
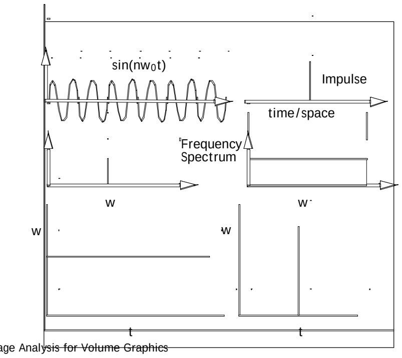


Image Example

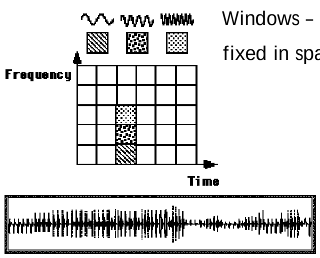


- 8x8 Blocked Window (Cosine) Transform
- Each DCT basis waveform represents a fixed frequency in two orthogonal directions
- Frequency spacing in each direction is an integer multiple of a base frequency

Time Frequency Diagram



Windowing & Filtering

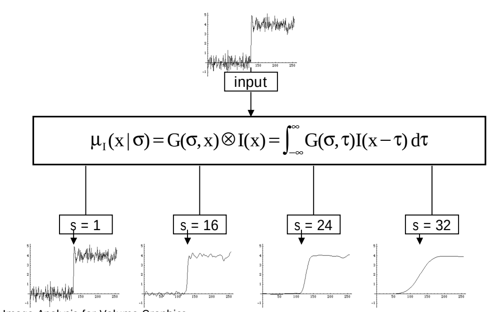


Windows –

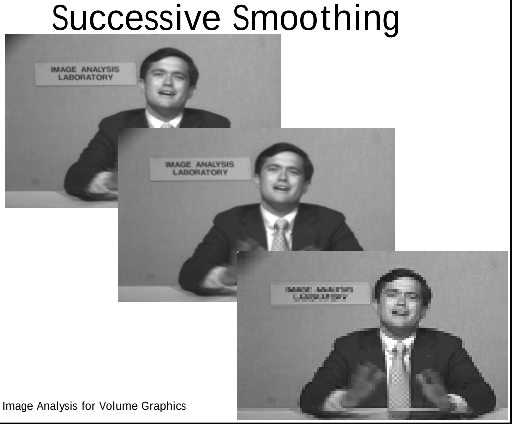
fixed in space and frequencies

Cannot resolve all features at all instants

Linear Scale Space



Successive Smoothing



Sub-sampled Images

- Keep 1 of 4 values from 2x2 blocks
- This naive approach introduces aliasing
- Sub-samples are bad representatives of area
- Little spatial correlation

Image Pyramid

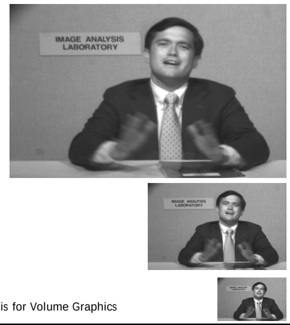
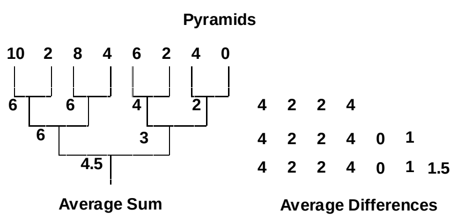


Image Pyramid – MIP MAP

- Average over a 2x2 block
- This is a rather straight forward approach
- This reduces aliasing and is a better representation
- However, this produces 11% expansion in the data

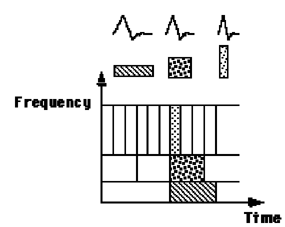
Image Pyramid – Another Twist



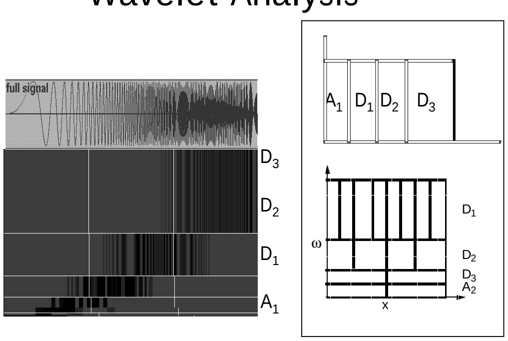
This is the Haar Wavelet Transform

Ideally !

Create new signal G such that ||F-G|| = e



Wavelet Analysis



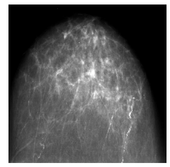
Why Wavelets? Because …

- We need to develop techniques to analyze data better through noise discrimination
- Wavelets can be used to detect features and to compare features
- Wavelets can provide compressed representations
- Wavelet Theory provides a unified framework for data processing

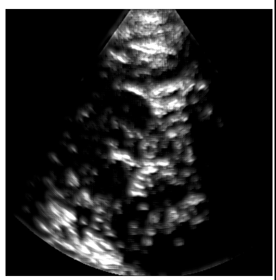
Scale-Coherent Structures

- Coherent structure - frequencies at all scales
- Examples - edges, peaks, ridges
- Locate extent and assign saliency

Wavelets – Analysis

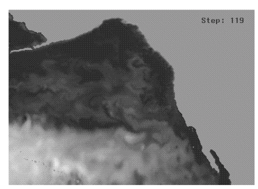


Wavelets – DeNoising

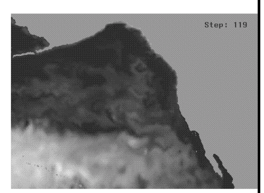


Wavelets – Compression

Original

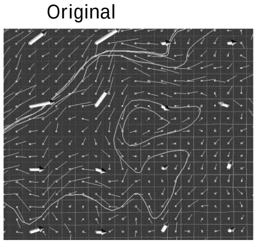


50:1

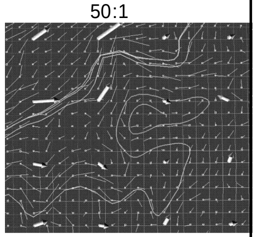


Wavelets – Compression

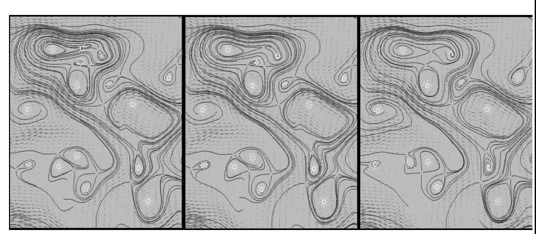
Original



50:1



Wavelets – Compression

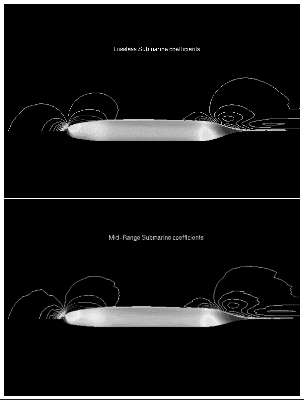


1/1 1/4 1/16

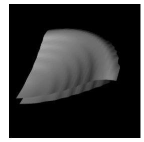
Yet Another Example

50%

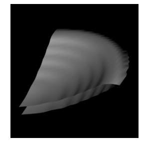
7%



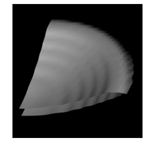
Final Example



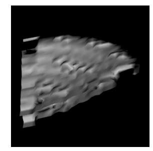
100%



50%

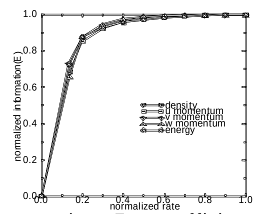


2%



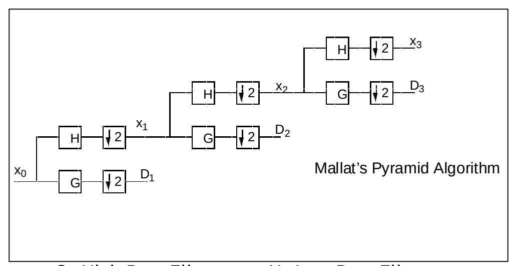
1%

Information Rate Curve



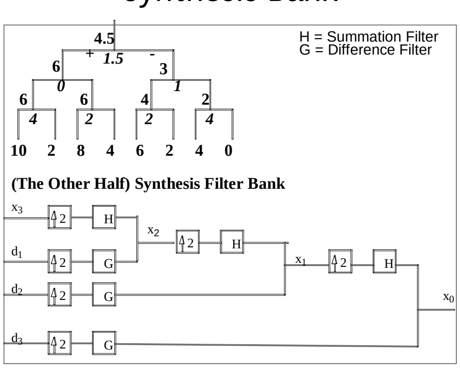
- Energy Compaction – Few coefficients can efficiently represent functions
- The Curve should be as vertical as possible near 0 rate

Filter Bank Implementation

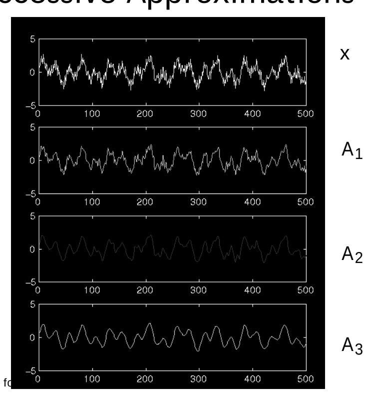


G: High Pass Filter H: Low Pass Filter

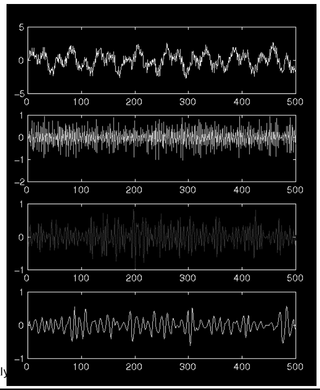
Synthesis Bank



Successive Approximations



Successive Details



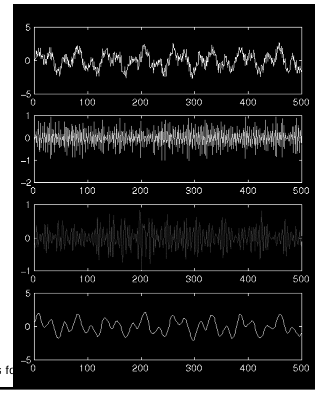
x

D_1

D_2

D_3

Wavelet Representation



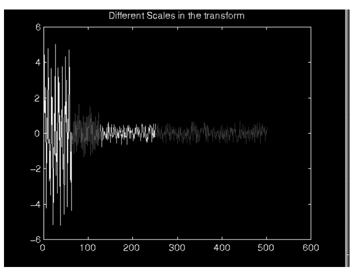
x

D_1

D_2

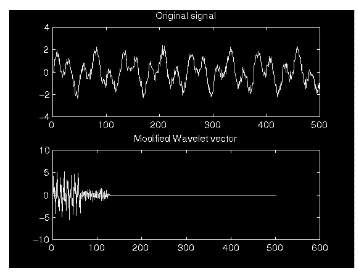
A_2

Coefficients



A_3 D_3 D_2 D_1

Lossey Compression



Lossey Compression

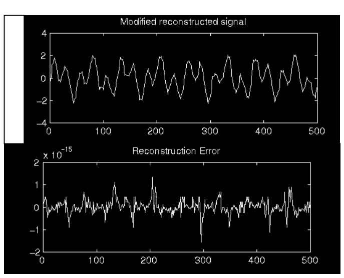


Image Example

A Frame

Another Frame

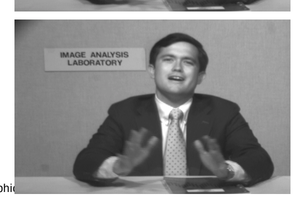
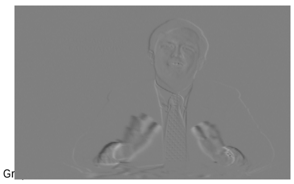


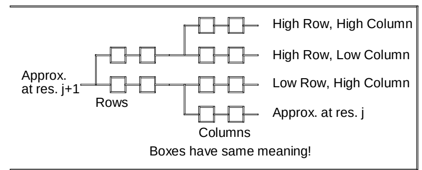
Image Example

Average

Difference



Wavelet Transform



Frequency Support

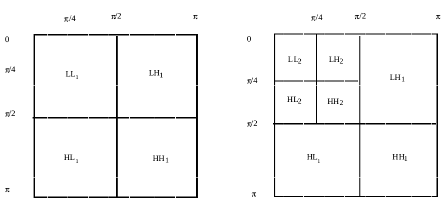


Image Example

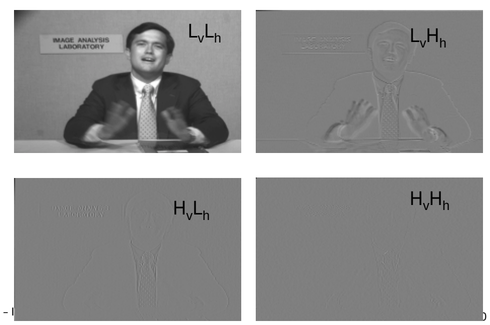
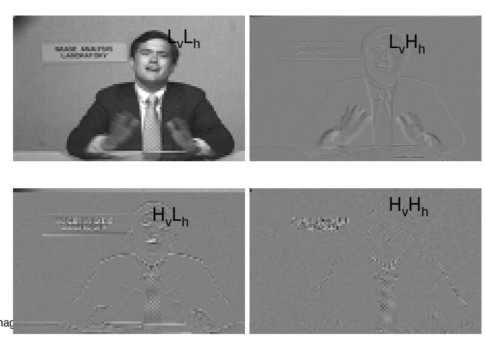


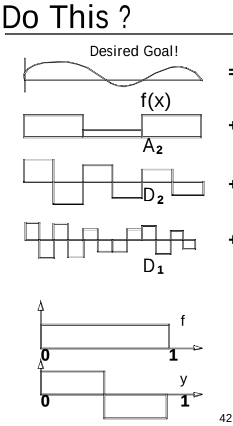
Image Example



How Does One Do This ?

$$\phi(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

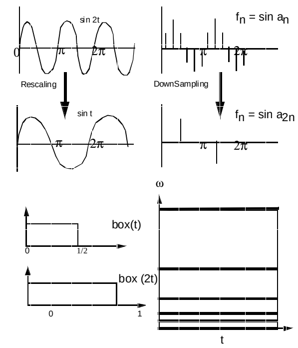
$$\psi(x) = \begin{cases} -1 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



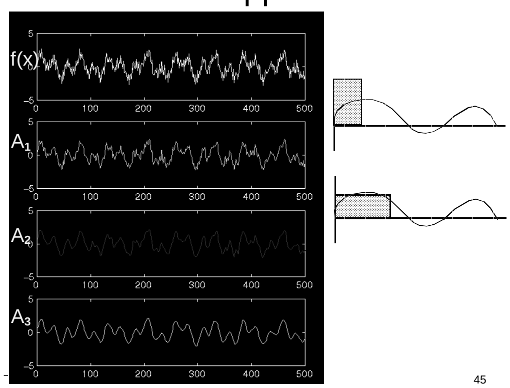
Dilations

- Rescaling Operation t --> 2t
- Down Sampling, n --> 2n
- Halve function support; Double frequency content
- Octave division of spectrum- Gives rise to different scales and resolutions
- Mother wavelet! - basic function gives rise to differing versions $\phi_j(x) = \frac{1}{2^{\frac{j}{2}}}\phi\left(\frac{x}{2^j}\right)$

Dilations

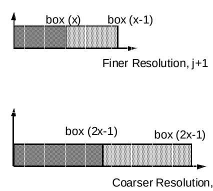


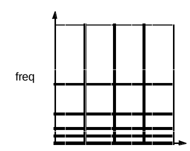
Successive Approximations



Translations

- Covers space-frequency diagram
- Versions are $\phi_{jk}(x) = \frac{1}{2^{\frac{j}{2}}}\phi\left(\frac{x-k}{2^j}\right)$

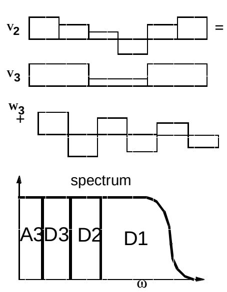




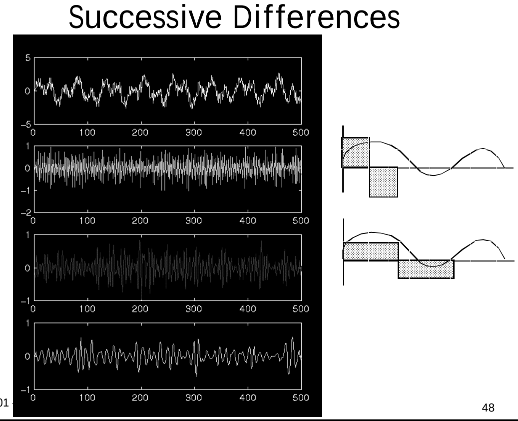
Wavelet Decomposition

- Induced functional Space - W_j.
- Related to V_js $V_j = V_{j+1} \oplus W_{j+1}$
- Space W_{j+1} is orthogonal to V_{j+1}
- *Also* $W_{j+1} \subset V_j$
- *J*-level wavelet decomposition -

$$V_j = V_{j+2} \oplus W_{j+2} \oplus W_{j+1}$$
$$V_0 = V_J + W_J + W_{J-1} + W_{J-2} + \ldots + W_1$$



Successive Differences



Wavelet Expansion

- Wavelet expansion (Tiling- j: scale, k: translates), Synthesis

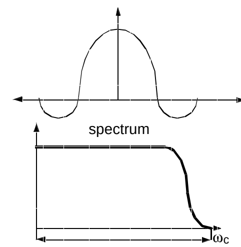
$$f(x) = \sum_k a_{Jk}\phi_{Jk}(x) + \sum_{j=1}^{J}\sum_k d_{jk}\psi_{jk}(x)$$

- Orthogonal transformation, Coarsest level of resolution - J
- Smoothing function - f, Detail function - y
- Analysis: $a_{Jk} = \int_{-\infty}^{\infty} f(x)\phi_{Jk}(x)dt$ $d_{jk} = \int_{-\infty}^{\infty} f(x)\psi_{jk}(x)dt$
- Commonly used wavelets are Haar, Daubechies and Coiflets

Scaling Functions

- Compact support
- Bandlimited - cut-off frequency
- Cannot achieve both
- DC value (or the average) is defined $\int\phi(x)dx = 1$
- Translates of f are orthogonal

$$\int\phi(x)\phi(x-k)dx = \delta(k)$$



Scaling Functions

- Nested smooth spaces

$$L^2 \supset V_0 \supset V_1 \supset V_2 \supset \ldots$$

- *Dilation Equation* - Haar

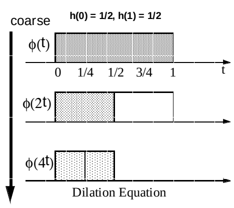
$$\phi(t) = 2h(0)\phi(2t) + 2h(1)\phi(2t-1)$$

- Generally –

$$\phi(t) = \sum\sqrt{2}h_k\phi(2t-k)$$

- Frequency Domain

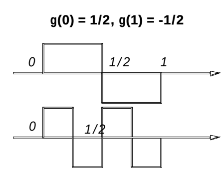
$$\hat{\Phi}(\omega) = \prod_{1}^{\infty}\left\{\frac{1}{\sqrt{2}}\hat{H}\left(\frac{\omega}{2}\right)\right\}\hat{\Phi}(0)$$



Wavelet Functions

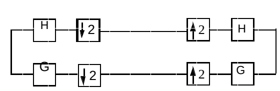
- Wavelet Equation - Haar System: G Filter

$$\psi(t) = 2g(0)\phi(2t) - 2g(1)\phi(2t-1)$$

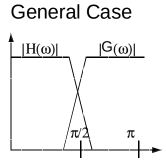


Perfect Reconstruction

- Synthesis and Analysis Filter Banks
- Synthesis Filters - Transpose of Analysis filters $\sum h(n) = \sqrt{2}$



General Case



Orthogonal Filter Banks

- Alternating Flip $g(k) = (-1)^k h(N-k)$
- Not symmetric - *h* is even length!
- Example $H = (h_0, h_1, h_2, h_3)$ $H^T = (h_3, h_2, h_1, h_0)$

$$G = (h_3, -h_2, h_1, -h_0) \quad G^T = (-h_0, h_1, -h_2, h_3)$$

- Orthogonality conditions

$$\sum h^2(n) = \delta(k) \qquad \sum h(n)g(n-2k) = 0$$

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2 \qquad \left|H\left(\frac{\pi}{2}\right)\right| = 1$$

Examples

$$h(0) + h(1) = \sqrt{2}$$
$$h^2(0) + h^2(1) = 1$$
$$h(0) = \frac{1}{\sqrt{2}} \qquad h(1) = \frac{1}{\sqrt{2}}$$

Haar

$$h(0) + h(1) + h(2) + h(3) = \sqrt{2}$$
$$h^2(0) + h^2(1) + h^2(2) + h(3) = 1$$
$$h(0)h(2) + h(1)h(3) = 0$$
$$h(0) = \frac{1+\sqrt{3}}{4\sqrt{2}} \quad h(1) = \frac{3+\sqrt{3}}{4\sqrt{2}}$$
$$h(2) = \frac{3-\sqrt{3}}{4\sqrt{2}} h(0) = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

Daubechies(2)

Approximation: Vanishing Moments Property

•Function is smooth - Taylor Series expansion

$$f(x) = \sum_{p=0} f^{(p)}(0)\frac{x^p}{p!}$$

•Wavelets with m vanishing moments

$$W[f(x);\psi] = \sum_{p=(m+1)} f^{(p)}(0)\frac{t^p}{p!}$$

•*Function with m derivatives can be accurately represented*!

Design of Compact Orthogonal Wavelets

•Compute scaling function

•Use *Refinement Equation* $\widehat{\Phi}(\omega) = \prod_{1}^{\infty}\left\{\frac{1}{\sqrt{2}}\widehat{H}\left(\frac{\omega}{2}\right)\right\}\widehat{\phi}(0)$

•N vanishing moments property - H(w) has a zero of order N at w=p

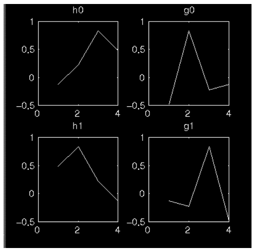
$$\widehat{H}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^p \widehat{Q}(\omega)$$

$$\widehat{Q}(\omega) = P\left(\sin\left(\frac{\omega}{2}\right)^2\right)$$

•$P(y)$ is p^{th} order polynomial (Daubechies 1992)

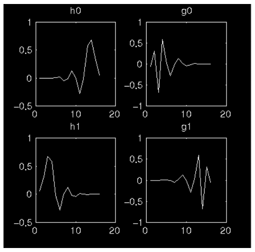
•Maxflat filter

Example



N=4

Example



N=16

Noise

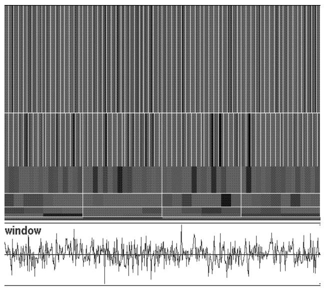
•Uncorrelated Gaussian noise is correlated

•Region of correlation is small at coarse scale

•Smooth versions - no noise

•Orthogonal transform - uncorrelated

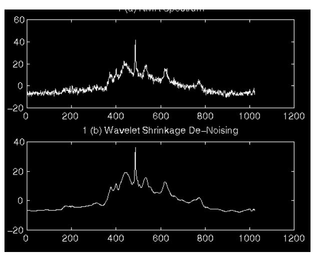
Noise Across Scales



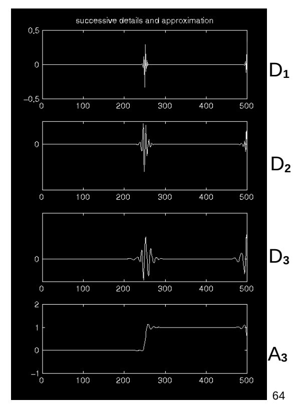
Denoising

- Statistical thresholding methods [Donohoe]
- Assuming Gaussian Noise
- Universal Threshold $\delta = 2\sigma\sqrt{\log(n)}$
- Smoothness guaranteed
- Hard $y_{hard}(t) = x(t) \qquad |x(t)| > \delta$
- Soft $y_{soft}(t) = \operatorname{sgn}(x(t))|x(t) - \delta| \quad |x(t)| > \delta$
- Works for additive noise since wavelet transform is linear $W(a,b)[f+\eta;\psi] = W(a,b)[f;\psi] + W(a,b)[\eta;\psi]$

Denoising Example



Discontinuity

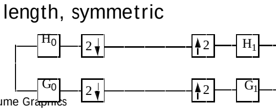


Multi-scale Edges

- Mallat and Hwang
- Location - maximas (edges) of wavelet coefficients at all scales
- Maxima chains for each edge
- Ranking - compute *Lipschitz* coefficient at all points
- Representation - store maximas
- Reconstruction- approximate but works in practice

Bi-Orthogonal Filter Banks

- Analysis/synthesis different
- Aliasing - overlap in spectras
- Alias cancellation $H_1(\omega)H_0(\omega+\pi) + G_1(\omega)G_0(\omega+\pi) = 0$
- Distortion Free (phase shift *l*) $H_1(\omega)H_0(\omega) + G_1(\omega)G_0(\omega) = 2e^{-j\omega l}$
- Alternating Flip condition valid
- Can be odd length, symmetric



Bi-Orthogonal Wavelets

•Governing equations

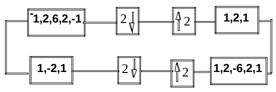
$$g(n) = (-1)^n h_1(N-n)$$

$$g_1(n) = (-1)^n h(N-n)$$

$$\sum_n h(n) h_1(n+2k) = \delta(k)$$

•Spline Wavelets - Many choices of either H_0 or H_1

•Choose H_0 as spline and solve equations to generate H_1



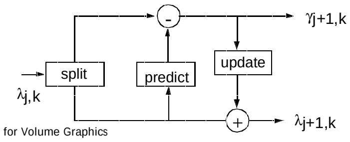
Bi-orthogonal: Lifting Scheme

•Lazy wavelet transform: split data in 2 parts

•Keep even part; predict (linear/cubic) odd part

•Lifting - update λ_{j+1} with γ_{j+1}: Maintain properties (moments, avg.)

•Synthesis is just flip of analysis



Summary

- Wavelets have good representation property
- They improve on image pyramid schemes
- Orthogonal and biorthogonal filter bank implementations are efficient
- Wavelets can filter signals
- They can efficiently denoise signals
- The presence of singularities can be detected from the magnitude of wavelet coefficients and their behavior across scales